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## Formalism for dilepton production via virtual photon bremsstrahlung in hadronic reactions

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### Abstract

We derive a set of new formulas for various distributions in dilepton production via virtual photon bremsstrahlung from pseudoscalar mesons and unpolarized spin-one-half fermions. These formulas correspond to the leading and sub-leading terms in the Low–Burnett–Kroll expansion for real photon bremsstrahlung. The relation of our leading-term formulas to previous works is also shown. Existing formulas are examined in the light of Lorentz covariance and gauge invariance. Numerical comparison is made in a simple example, where an “exact” formula and real photon data exist. The results reveal large discrepancies among different bremsstrahlung formulas. Of all the leading-term bremsstrahlung formulas, the one derived in this work agrees best with the exact formula. The issues of  $M_T$ -scaling and event generators are also addressed.

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## I. INTRODUCTION

Even if electromagnetic phenomena rank among the best understood in particle physics, their merging with hadronic processes often brings ambiguities that cannot be resolved on a purely theoretical basis. It is generally believed, however, that many of the reactions in which photons and dileptons are produced can be described as bremsstrahlung from incoming and outgoing charged particles, at least within a limited kinematic range.

It is well known that the cross section for production of photons with very low energies is uniquely determined by the cross section of the corresponding nonradiative reaction [1–6]. In the dilepton sector, the situation is less clear. Several different bremsstrahlung formulas have been proposed [7–14], which, as will be demonstrated in this work, do not agree with one another very well. Some of these formulas fail to satisfy constraints implied by general principles such as Lorentz covariance and gauge invariance. Besides the formulas cited above, a few additional formulas exist that have been designed for specific kinematic regions [15,16]. They will not be considered here.

The purpose of this work is to present a set of consistent formulas for various distributions in dilepton bremsstrahlung from pseudoscalar mesons and unpolarized spin-one-half fermions. We consider the terms that are proportional to the square of the nonradiative matrix element (leading term approximation) and its derivatives (sub-leading approximation). We first derive the formula for the most general quantity, namely the double differential cross section in the momenta of leptons. The correct form for it has not been yet known even in the leading term approximation. Then we arrive at the cross section in dilepton mass and momentum. Its leading part differs only by higher-order terms in the dilepton four-momentum from one of the already existing formulas [12].

To stress the importance of using the correct virtual bremsstrahlung formalism, we compare our formulas to those already existing in the literature. The comparison is made first on general grounds and is followed by an application of the formulas to a simple physical process. All formulas are scrutinized from the point of view of Lorentz covariance [17] and from what we will call a global variable test.

The global variable test is based on the finding [18,19] that in a one-photon approximation, gauge invariance leads to the following relation for the inclusive<sup>1</sup> differential cross section in global dilepton quantities  $M$  (dilepton mass) and  $\mathbf{q}$  (dilepton momentum)

$$\frac{d^4\sigma^{e^+e^-}}{dM^2d^3q} = \mathcal{T}(M^2) \left( \frac{d^3\sigma^{\gamma^*}}{d^3q} \right)_M \quad (1.1)$$

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<sup>1</sup>We will omit the word inclusive in what follows. All our cross sections for photon or dilepton production are semi-inclusive (integrated over the final state hadron momenta in a reaction with given number and types of final state particles). Some of the relations are more general and hold also for inclusive cross sections, which are given as sums of the semi-inclusive cross sections over all possible reactions with chosen initial particles [as, e.g., (1.1), which is valid even for exclusive cross sections]. In some cases, the sum over all possible reactions should be supplemented to make the relation inclusive.

with the function  $\mathcal{T}$  given by

$$\mathcal{T}(M^2) = \frac{\alpha}{3\pi} \frac{1}{M^2} \left( 1 + \frac{2\mu^2}{M^2} \right) \sqrt{1 - \frac{4\mu^2}{M^2}} , \quad (1.2)$$

where  $\alpha$  is the fine structure constant and  $\mu$  the lepton mass. The rightmost quantity in Eq. (1.1) is called the cross section for virtual photon production. It does not have a direct physical meaning, as it is not experimentally accessible. Nevertheless, Eqs. (1.1) and (1.2) can test the soundness of theoretical formulas, because they show the only two places in which the lepton mass  $\mu$  may and must (unless neglected) appear. From a technical point of view, the virtual photon cross section is calculable more easily than the dilepton cross section [18]. Of course, if one needs the distribution in the momenta of leptons, a more involved approach is unavoidable.

Furthermore, if an otherwise identical reaction exists in which a photon is produced instead of a dilepton (this need not always be the case, viz.,  $\pi^+\pi^- \rightarrow e^+e^-$ ), the relation

$$\lim_{M \rightarrow 0} \left( \frac{d^3\sigma^{\gamma^*}}{d^3q} \right)_M = \frac{d^3\sigma^\gamma}{d^3q} \quad (1.3)$$

must be fulfilled [18] with the differential cross section for real photon production on the right-hand side.

Relations analogous to (1.1) and (1.3) can also be written for the corresponding quantities in other than two-body reactions, namely for decays and processes with more than two particles in the initial state [20].

In the next section, we derive the formulas appropriate for the production of very soft (low mass, low momentum) dileptons via virtual photon bremsstrahlung in reactions with charged pseudoscalar particles. Section III deals with the virtual bremsstrahlung from fermions. In Sec. IV, we show the additional approximations which lead to the various formulas that have appeared explicitly or as a part of more complex expressions in the literature. All the formulas are examined to see if they fulfill the Lorentz covariance and global variable tests. In Sec. V, we introduce a simple theoretical model of the process  $\rho^0 \rightarrow \pi^+\pi^-e^+e^-$  (which has not been experimentally investigated yet) based on a successful description [21] of the recently observed [22] decay  $\rho^0 \rightarrow \pi^+\pi^-\gamma$ . The former process will be a testing ground for various virtual bremsstrahlung formulas in Sec. VI with the theoretical distribution from Sec. V serving as a reference. We summarize our main points and add a few comments in Sec. VII. Some related issues are discussed in the Appendices. In Appendix A we show how  $M_T$ -scaling transpires from the leading term virtual bremsstrahlung formalism. Appendix B deals with the “exact” formula for the  $\rho^0 \rightarrow \pi^+\pi^-\gamma^*$  branching ratio. Appendix C addresses the issue of photon and dilepton event generators that conserve energy and momentum.

## II. DILEPTONS FROM VIRTUAL BREMSSTRAHLUNG OFF PSEUDOSCALAR MESONS

In this section, we assume that all charged particles are pseudoscalar mesons. The technically more involved case of virtual bremsstrahlung from spin-one-half fermions will be treated in the next section.

### A. Leading term approximation

Let us consider a  $2 \rightarrow n$  hadronic reaction

$$a + b \rightarrow 1 + 2 + \cdots + n \quad (2.1)$$

and denote its matrix element as  $\mathcal{M}_0 \equiv \mathcal{M}_0(p_a, p_b, p_1, \cdots, p_n)$ . Our aim is to find the matrix element  $\mathcal{M}$  of the reaction

$$a + b \rightarrow 1 + 2 + \cdots + n + l^+ + l^-, \quad (2.2)$$

in which a soft lepton pair with the four-momentum  $q = p_+ + p_-$  is produced in addition to  $n$  hadrons. The dominant contribution to the matrix element  $\mathcal{M}$  comes from diagrams where the virtual photon is attached to one of the external legs (see Fig. 1). The diagrams in which a virtual photon is radiated from internal lines give subleading contributions. This is caused by the nonvanishing virtuality contributions of the type  $(p^2 - m^2)$  to the denominators of newly emerging propagators. Such terms do not appear if one of the two particles attached to the electromagnetic vertex is real ( $p^2 = m^2$ ). See below.

Using the Feynman rules of pseudoscalar electrodynamics [23] we can immediately write down the contribution to the matrix element  $\mathcal{M}$  from radiation of an initial ( $x = a, b$ )

$$\mathcal{M}_x = -eQ_x \mathcal{M}_0 \frac{(2p_x - q)^\mu}{2p_x \cdot q - M^2} L_\mu \quad (2.3)$$

and a final ( $i = 1, \cdots, n$ )

$$\mathcal{M}_i = eQ_i \mathcal{M}_0 \frac{(2p_i + q)^\mu}{2p_i \cdot q + M^2} L_\mu \quad (2.4)$$

state particle. Above,  $e$  is the positive elementary charge,  $Q_x$  ( $Q_i$ ) is the charge of an initial (a final) particle, and  $M$  is the dilepton mass ( $q^2 = M^2$ ). As is customary in this field, we have supposed that the nonradiative matrix element does not change when an incoming or outgoing momentum becomes “slightly” off-mass-shell; for example,

$$\mathcal{M}_0(p_a - q, p_b, p_1, \cdots, p_n) = \mathcal{M}_0(p_a, p_b, p_1, \cdots, p_n) \equiv \mathcal{M}_0. \quad (2.5)$$

The lepton part is given by

$$L_\mu = \frac{e}{M^2} \bar{u}^{(s-)}(p_-) \gamma_\mu v^{(s+)}(p_+). \quad (2.6)$$

With this choice of  $L_\mu$ , the matrix element for photon production is obtained by the substitution  $L_\mu \rightarrow \epsilon_\mu$ , where  $\epsilon$  is the photon polarization vector. Summing the contributions of individual diagrams, we arrive at

$$\mathcal{M} = e \mathcal{M}_0 J \cdot L \quad (2.7)$$

with

$$J^\mu = - \sum_{x=a,b} Q_x \frac{(2p_x - q)^\mu}{2p_x \cdot q - M^2} + \sum_{i=1}^n Q_i \frac{(2p_i + q)^\mu}{2p_i \cdot q + M^2} . \quad (2.8)$$

The four-vector  $J$  satisfies the important relation

$$J \cdot q = -Q_a - Q_b + \sum_{i=1}^n Q_i = 0 , \quad (2.9)$$

which reflects charge conservation. Squaring the matrix element (2.7) and summing over the spins of leptons, we get

$$\sum_{s_+, s_-} |\mathcal{M}|^2 = 4\pi\alpha |\mathcal{M}_0|^2 J^\mu J^\nu L_{\mu\nu} . \quad (2.10)$$

The tensor  $L_{\mu\nu}$  is defined by

$$L_{\mu\nu} = \sum_{s_+, s_-} L_\mu L_\nu^* . \quad (2.11)$$

A straightforward calculation leads to

$$L_{\mu\nu} = \frac{8\pi\alpha}{M^4} (q_\mu q_\nu - l_\mu l_\nu - M^2 g_{\mu\nu}) , \quad (2.12)$$

where we have introduced the four-vector  $l = p_+ - p_-$  as the difference of leptons' momenta. Using the above relation we can rewrite (2.10) in the form

$$\sum_{s_+, s_-} |\mathcal{M}|^2 = |\mathcal{M}_0|^2 \frac{32\pi\alpha^2}{M^2} \left[ -J^2 - \frac{1}{M^2} (l \cdot J)^2 \right] . \quad (2.13)$$

Inserting this into the relation for the unpolarized cross section of the reaction (2.2) leads to

$$\begin{aligned} d\sigma &= \frac{1}{4E_a E_b |\mathbf{v}_a - \mathbf{v}_b|} |\mathcal{M}_0|^2 (2\pi)^4 \delta(p_a + p_b - \sum_i p_i - q) \\ &\times \frac{32\pi\alpha^2}{M^2} \left[ -J^2 - \frac{1}{M^2} (l \cdot J)^2 \right] \prod_{i=1}^n \frac{d^3 p_i}{2E_i (2\pi)^3} \prod_{l=+,-} \frac{d^3 p_l}{2E_l (2\pi)^3} . \end{aligned} \quad (2.14)$$

After neglecting the dilepton four-momentum in the argument of the  $\delta$ -function and integrating over the momenta of final hadrons, we get

$$E_+ E_- \frac{d^6 \sigma^{e^+ e^-}}{d^3 p_+ d^3 p_-} = \frac{\alpha^2}{8\pi^4} \frac{1}{M^2} \int \left[ -J^2 - \frac{1}{M^2} (l \cdot J)^2 \right] d\sigma_0 , \quad (2.15)$$

where

$$d\sigma_0 = \frac{1}{4E_a E_b |\mathbf{v}_a - \mathbf{v}_b|} |\mathcal{M}_0|^2 (2\pi)^4 \delta(p_a + p_b - \sum_i p_i) \prod_{i=1}^n \frac{d^3 p_i}{2E_i (2\pi)^3} \quad (2.16)$$

is the infinitesimal cross section of the reaction (2.1). Double differential cross section  $E_+ E_- d^6 \sigma^{e^+ e^-} / d^3 p_+ d^3 p_-$  is the most general quantity that characterizes the production of pairs of unpolarized unlike-sign leptons. Knowing it, we can find any other distribution (but not *vice versa*).

A little exercise from relativistic kinematics provides us with the following general formula

$$E \frac{d^6 \sigma^{e^+ e^-}}{dM^2 d^3 q d\tilde{\Omega}_+} = \frac{1}{4} \sqrt{1 - \frac{4\mu^2}{M^2}} E_+ E_- \frac{d^6 \sigma^{e^+ e^-}}{d^3 p_+ d^3 p_-} , \quad (2.17)$$

where  $E = E_+ + E_-$  is the dilepton energy and  $d\tilde{\Omega}_+$  is the solid angle element for positron momentum in the dilepton rest frame. Using (2.15) and (2.17), we arrive at another formula of our virtual bremsstrahlung formalism

$$E \frac{d^6 \sigma^{e^+ e^-}}{dM^2 d^3 q d\tilde{\Omega}_+} = \frac{\alpha^2}{32\pi^4} \frac{1}{M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} \int \left[ -J^2 - \frac{1}{M^2} (l \cdot J)^2 \right] d\sigma_0 . \quad (2.18)$$

Finally, integrating over the positron angles in the dilepton rest frame, we obtain the differential cross section in global dilepton variables

$$E \frac{d^4 \sigma^{e^+ e^-}}{dM^2 d^3 q} = \frac{\alpha^2}{12\pi^3} \frac{1}{M^2} \left( 1 + \frac{2\mu^2}{M^2} \right) \sqrt{1 - \frac{4\mu^2}{M^2}} \int (-J^2) d\sigma_0 . \quad (2.19)$$

This is just formula (1.1) with the virtual photon cross section given by

$$E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{\alpha}{4\pi^2} \int (-J^2) d\sigma_0 . \quad (2.20)$$

Sometimes it is advantageous to express  $J^2$  in terms of three-dimensional vectors. This can easily be done [12] using the relation  $J_0 = E^{-1} \mathbf{J} \cdot \mathbf{q}$ , which follows from Eq. (2.9). We thus get

$$-J^2 = (\mathbf{J} \times \mathbf{n})^2 + \frac{M^2}{E^2} (\mathbf{J} \cdot \mathbf{n})^2 , \quad (2.21)$$

where  $\mathbf{n} = \mathbf{q}/|\mathbf{q}|$  is the unit vector in the dilepton momentum direction.

To investigate the limit (1.3) of (2.20), it is sufficient to realize that due to (2.21), the part of  $J^\mu$  that is proportional to  $q^\mu$  will not contribute in the limit  $M \rightarrow 0$ . What remains can be written as

$$\omega \frac{d^3 \sigma^\gamma}{d^3 q} = \frac{\alpha}{4\pi^2} \int (-J_R^2) d\sigma_0 \quad (2.22)$$

with  $\omega = |\mathbf{q}|$  and

$$J_R = \sum Q'_i \frac{p_i}{p_i \cdot q} = \frac{1}{\omega} \sum \frac{Q'_i}{E_i} \frac{p_i}{1 - \mathbf{v}_i \cdot \mathbf{n}} . \quad (2.23)$$

To make the formula more compact, we have introduced the variable  $Q'_i$ , which is identical with the charge of final particles and acquires the opposite sign for initial particles. The unspecified sum runs over all (initial and final) hadrons and  $\mathbf{v}_i = \mathbf{p}_i/E_i$  is the velocity of the  $i$ th hadron. Eqs. (2.22) and (2.23) combine to give the well known leading term formula for real photon bremsstrahlung [24,25].

The central results of this subsection are the formulas (2.15), (2.18), and (2.19) for various differential cross sections of dilepton production via virtual photon bremsstrahlung in the leading term approximation.

## B. Next-to-leading term approximation

When going beyond the next-to-leading order, the nonradiative matrix element is no longer considered to be immune against the changes in incoming and outgoing momenta. We write, instead of Eq. (2.5),

$$\mathcal{M}_0(p_x - q) = \mathcal{M}_0 - q^\alpha \frac{\partial \mathcal{M}_0}{\partial p_x^\alpha} \quad (2.24)$$

if one of the incoming momenta changes, and

$$\mathcal{M}_0(p_i + q) = \mathcal{M}_0 + q^\alpha \frac{\partial \mathcal{M}_0}{\partial p_i^\alpha} \quad (2.25)$$

to account for a change in one of the outgoing momenta. In the above two equations, we suppress the momenta that keep their “nonradiative” values. If we now simply incorporated the “corrected” values (2.24) and (2.25) into the expressions (2.3) and (2.4) for the radiative matrix elements, and summed these up, we would obtain a non-gauge-invariant quantity. After replacing the four-vector  $L_\mu$  by the virtual photon four-momentum, it would not vanish. The reason is that we have not yet included the “contact” terms, which are generated from the strong interaction Lagrangian by the minimal electromagnetic interaction principle. The latter says that the electromagnetic interaction terms appear as a result of the substitution

$$p^\alpha \rightarrow p^\alpha - eQg^{\alpha\mu} , \quad (2.26)$$

where the index  $\mu$  is to be contracted with the real photon polarization vector or the virtual photon propagator. To find the contact terms in our case, we make a formal expansion of  $\mathcal{M}_0(p_i^\alpha - eQ_i g^{\alpha\mu})$ , where  $p_i$  denotes any of the incoming or outgoing momenta. We thus find that the contact term associated with the  $i$ th hadron is

$$C_i^\mu = -eQ_i \frac{\partial \mathcal{M}_0}{\partial p_{i\mu}} . \quad (2.27)$$

For the same reason as stated in the previous subsection, the radiation from internal lines will not contribute in this approximation either. Putting it all together, the radiative matrix element comes out to be

$$\mathcal{M} = e \, K \cdot L \, , \quad (2.28)$$

where  $L$  is the four-vector defined by Eq. (2.6), and

$$K^\mu = \mathcal{M}_0 \, J^\mu + \sum_i Q_i \frac{\partial \mathcal{M}_0}{\partial p_i^\alpha} \left( \frac{p_i^\mu q^\alpha}{p_i \cdot q} - g^{\mu\alpha} \right). \quad (2.29)$$

We retained only the leading and next-to-leading terms in  $q$ . The sum runs over both incoming and outgoing hadrons and the four-vector  $J$  is given by Eq. (2.8). To get the cross section for producing a pair of unpolarized leptons, we need the quantity

$$\sum_{s_+, s_-} |\mathcal{M}|^2 = 4\pi\alpha \, H^{\mu\nu} L_{\mu\nu} \, , \quad (2.30)$$

with symmetric tensor  $L_{\mu\nu}$  defined by (2.11), and

$$H^{\mu\nu} = \frac{1}{2} (K^\mu K^{*\nu} + K^{*\mu} K^\nu) \, . \quad (2.31)$$

Keeping only the  $q$ -terms of the same order as before, we easily obtain

$$\begin{aligned} H^{\mu\nu} = |\mathcal{M}_0|^2 \, J^\mu J^\nu + \frac{1}{2} \sum_{i,j} \frac{Q_i Q'_j}{(p_i \cdot q)(p_j \cdot q)} \frac{\partial |\mathcal{M}_0|^2}{\partial p_i^\beta} p_{i\alpha} \\ \times \left[ p_j^\mu (g^{\nu\alpha} q^\beta - g^{\nu\beta} q^\alpha) + p_j^\nu (g^{\mu\alpha} q^\beta - g^{\mu\beta} q^\alpha) \right] \, . \end{aligned} \quad (2.32)$$

To simplify the formula, we have again used the convention  $Q'_j = -Q_j$  for the incoming hadrons and  $Q'_j = Q_j$  for the outgoing hadrons. The differential cross section in lepton pair momenta now reads

$$\begin{aligned} E_+ E_- \frac{d^6 \sigma^{e^+ e^-}}{d^3 p_+ d^3 p_-} = \frac{\alpha^2}{8\pi^4} \frac{1}{M^2} \left\{ \int [-J^2 - \frac{1}{M^2} (l \cdot J)^2] d\sigma_0 + \sum_{i,j} \int \frac{Q_i Q'_j}{M^2 (p_i \cdot q)(p_j \cdot q)} \right. \\ \left. \times \left[ p_i \cdot q (l \cdot p_j l^\beta + M^2 p_j^\beta) - (l \cdot p_i l \cdot p_j + M^2 p_i \cdot p_j) q^\beta \right] d\sigma_{0,i\beta} \right\} \, . \end{aligned} \quad (2.33)$$

The notation is same as we met in Eq. (2.15), except for

$$d\sigma_{0,i\beta} = \frac{1}{4E_a E_b |\mathbf{v}_a - \mathbf{v}_b|} \frac{\partial |\mathcal{M}_0|^2}{\partial p_i^\beta} (2\pi)^4 \delta(p_a + p_b - \sum_k p_k) \prod_{k=1}^n \frac{d^3 p_k}{2E_k (2\pi)^3} \, . \quad (2.34)$$

Using the same procedure as in subsection II A, we arrive at the differential cross section in global dilepton variables in the form (1.1) with the virtual photon cross section given by

$$\begin{aligned} E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{\alpha}{4\pi^2} \left\{ \int (-J^2) d\sigma_0 + \sum_{i,j} \int \frac{Q_i Q'_j}{(p_i \cdot q)(p_j \cdot q)} \right. \\ \left. \times \left[ (p_i \cdot q) p_j^\beta - (p_i \cdot p_j) q^\beta \right] d\sigma_{0,i\beta} \right\} \, . \end{aligned} \quad (2.35)$$

We will comment on the zero photon mass limit of this equation in Sec. III.



### C. Decays

The description of virtual bremsstrahlung in hadronic decays can be achieved following the lines sketched for the two-body reactions. We need only to change the number of incoming particles to one and replace the cross sections by decay widths. This follows from the similar structure of the relations between cross section, or decay width, on the one hand and the matrix element squared on the other. Another important factor is the universality (with respect to the numbers of incoming and outgoing particles) of Eqs. (2.7) and (2.8).

Let us consider the decay

$$a \rightarrow 1 + 2 + \cdots + n + l^+ + l^- . \quad (2.36)$$

For its differential decay width we can write, in the leading term approximation,

$$E_+ E_- \frac{d^6 \Gamma^{e^+ e^-}}{d^3 p_+ d^3 p_-} = \frac{\alpha^2}{8\pi^4} \frac{1}{M^2} \int \left[ -J^2 - \frac{1}{M^2} (l \cdot J)^2 \right] d\Gamma_0 . \quad (2.37)$$

The quantity

$$d\Gamma_0 = \frac{1}{2m_a} |\mathcal{M}_0|^2 (2\pi)^4 \delta(p_a - \sum_i p_i) \prod_{i=1}^n \frac{d^3 p_i}{2E_i (2\pi)^3} \quad (2.38)$$

is the invariant decay width into an infinitesimal element of the momentum space for the decay

$$a \rightarrow 1 + 2 + \cdots + n . \quad (2.39)$$

The other leading-term formulas can be easily modified as well. The differential decay width in global dilepton quantities is

$$E \frac{d^4 \Gamma^{e^+ e^-}}{dM^2 d^3 q} = \mathcal{T}(M^2) \frac{\alpha}{4\pi^2} \int (-J^2) d\Gamma_0 . \quad (2.40)$$

If a nonradiative decay (2.39) contains only two particles in the final state, we can proceed further to get

$$\frac{1}{\Gamma_0} \frac{d^2 \Gamma^{e^+ e^-}}{dM^2 dq^*} = \mathcal{T}(M^2) \frac{\alpha}{4\pi^2} \frac{q^{*2}}{E^*} \int (-J^2) d\Omega_{\mathbf{q}^*} , \quad (2.41)$$

where the asterisk refers to quantities in the rest frame of the parent particle and  $q^* = |\mathbf{q}^*|$ . For a hypothetical decay into two particles and a virtual photon it means

$$\frac{1}{\Gamma_0} \left( \frac{d\Gamma^{\gamma^*}}{dq^*} \right)_M = \frac{\alpha}{4\pi^2} \frac{q^{*2}}{E^*} \int (-J^2) d\Omega_{\mathbf{q}^*} . \quad (2.42)$$

### III. VIRTUAL BREMSSTRAHLUNG FROM FERMIONS

It has been shown in [12] that for virtual bremsstrahlung from fermions, the terms proportional to  $p^\mu$  in the expression (2.8) can be obtained in the same way as in the real photon bremsstrahlung off fermions (see, e.g., [25]). In order to obtain the form of  $J$  analogous to (2.8), additional approximations are required. One has to discard some terms while keeping others of the same order in  $q$ . The justification of such a procedure is unclear.

One may hope that if the summing (or averaging) over spins of hadrons is performed, the additional contributions will rearrange and the formulas identical to those pertaining to the pseudoscalar case will be restored. This would resemble a similar development in the next-to-leading terms in real photon bremsstrahlung [3].

To investigate such a possibility, let us assume that in the nonradiative reaction (2.1), a charged fermion-antifermion (e.g., proton-antiproton) pair is produced. To be more definite, we assign the index 1 to the fermion ( $Q_1 = 1$ ) and 2 to the antifermion ( $Q_2 = -1$ ). Now, the matrix element exhibits the form

$$\mathcal{M}_0 = \bar{u}^{(s_1)}(p_1) \Gamma v^{(s_2)}(p_2) , \quad (3.1)$$

where  $\Gamma \equiv \Gamma(p_a, p_b, p_1, \dots, p_n)$  is a matrix in the spinor space. To simplify notation, we will display its arguments only if they differ from the values just shown. Squaring the matrix element (3.1) and summing over the spin projections  $s_1$  and  $s_2$ , we get

$$|\overline{\mathcal{M}_0}|^2 = \text{Tr} [(\hat{p}_1 + m)\Gamma(\hat{p}_2 - m)\Gamma'] , \quad (3.2)$$

where  $\Gamma' = \gamma^0 \Gamma^\dagger \gamma^0$ . The quantity (3.2) determines the unpolarized cross section of the nonradiative reaction (2.1) with the two spin-one-half hadrons in the final state. For a later use, let us notice that

$$\frac{\partial |\overline{\mathcal{M}_0}|^2}{\partial p_1^\alpha} = \text{Tr} \left\{ \left[ \Gamma' \gamma_\alpha \Gamma + \Gamma'(\hat{p}_1 + m) \frac{\partial \Gamma}{\partial p_1^\alpha} + \frac{\partial \Gamma'}{\partial p_1^\alpha} (\hat{p}_1 + m) \Gamma \right] (\hat{p}_2 - m) \right\} , \quad (3.3)$$

and

$$\frac{\partial |\overline{\mathcal{M}_0}|^2}{\partial p_2^\alpha} = \text{Tr} \left\{ (\hat{p}_1 + m) \left[ \Gamma \gamma_\alpha \Gamma' + \frac{\partial \Gamma}{\partial p_2^\alpha} (\hat{p}_2 - m) \Gamma' + \Gamma(\hat{p}_2 - m) \frac{\partial \Gamma'}{\partial p_2^\alpha} \right] \right\} . \quad (3.4)$$

As a next step, let us consider the corresponding dilepton-producing reaction (2.2). We will concentrate on virtual bremsstrahlung from fermions and, for simplicity, take all the mesons neutral. In addition, we neglect any anomalous electromagnetic interactions. The changes in the strong interaction matrix element will be incorporated by

$$\Gamma(p_i + q) = \Gamma + q^\alpha \frac{\partial \Gamma}{\partial p_i^\alpha} . \quad (3.5)$$

The contact electromagnetic interaction term associated with  $i$ th fermion line leaving the strong interaction core comes out as

$$C_i^\mu = -eQ_i \frac{\partial \Gamma}{\partial p_{i\mu}} . \quad (3.6)$$

Using the Feynman rules for spinor electrodynamics, Dirac equation in momentum space, and the properties of the  $\gamma$ -matrices, we find the matrix element of reaction (2.2)

$$\mathcal{M} = eL_\mu \bar{u}^{(s_1)}(p_1) K^\mu v^{(s_2)}(p_2) . \quad (3.7)$$

The four-vector  $L_\mu$  has the same meaning as before [see (2.6)]. We have, neglecting higher than linear  $q$ -terms in the numerators,

$$K^\mu = \Gamma J^\mu + \frac{1}{4} \frac{1}{p_1 \cdot q} [\gamma^\mu, \hat{q}] \Gamma + \frac{1}{4} \frac{1}{p_2 \cdot q} \Gamma [\gamma^\mu, \hat{q}] + (g^{\mu\alpha} q^\beta - g^{\mu\beta} q^\alpha) \sum_{i=1,2} Q_i \frac{p_{i,\alpha}}{p_i \cdot q} \frac{\partial \Gamma}{\partial p_i^\beta} . \quad (3.8)$$

In accordance with (2.8), we denoted

$$J^\mu = \frac{(2p_1 + q)^\mu}{2p_1 \cdot q + M^2} - \frac{(2p_2 + q)^\mu}{2p_2 \cdot q + M^2} . \quad (3.9)$$

At this point we can clearly see the difference between pseudoscalar and fermion cases. When we neglected the changes in the strong interaction core as well as the contact terms in pseudoscalar case, we obtained immediately the leading term approximation in the form (2.7). For fermions, the extra terms (those with commutators) prevent us from reaching the same goal.

The sum over the spins of the matrix element (3.7) squared assumes the form

$$\sum_{s_1, s_2, s_+, s_-} |\mathcal{M}|^2 = 4\pi\alpha H^{\mu\nu} L_{\mu\nu} . \quad (3.10)$$

The tensor  $L_{\mu\nu}$  is given by Eq. (2.12). Up to the leading and next-to-leading order in  $q$ ,

$$\begin{aligned} H^{\mu\nu} = & \overline{|\mathcal{M}_0|^2} J^\mu J^\nu + \frac{1}{2} \left\{ J^\mu \text{Tr}[(\hat{p}_1 + m) \Gamma (\hat{p}_2 - m) K'^\nu] \right. \\ & \left. + J^\mu \text{Tr}[(\hat{p}_1 + m) K^\nu (\hat{p}_2 - m) \Gamma] + \mu \leftrightarrow \nu \right\} \end{aligned} \quad (3.11)$$

with  $K'^\nu = \gamma_0 (K^\nu)^\dagger \gamma_0$ . A straightforward manipulation guides us to an expression in parts of which we are able to identify the right-hand sides of Eqs. (3.3) and (3.4). After replacing them by corresponding derivatives we get

$$\begin{aligned} H^{\mu\nu} = & \overline{|\mathcal{M}_0|^2} J^\mu J^\nu + \frac{1}{2} \sum_{i,j} \frac{Q_i Q'_j}{(p_i \cdot q)(p_j \cdot q)} p_{i\alpha} \frac{\partial \overline{|\mathcal{M}_0|^2}}{\partial p_i^\beta} \\ & \times \left[ p_j^\mu (g^{\nu\alpha} q^\beta - g^{\nu\beta} q^\alpha) + p_j^\nu (g^{\mu\alpha} q^\beta - g^{\mu\beta} q^\alpha) \right] . \end{aligned} \quad (3.12)$$

This is identical with what we would get from Eq. (2.32) for two outgoing charged mesons, with only one difference. A simple square of nonradiative mesonic matrix element is replaced by the sum over fermion spins. As an independent check of our result, we explored also other situations (an incoming fermion-antifermion pair, one incoming–one outgoing

fermion/antifermion) and reached the same conclusion. For the initial state fermions, the sum is replaced by the average. The generalization to more than one fermion pair is obvious.

The tensor  $H^{\mu\nu}$  is the central object of all bremsstrahlung formulas. We have therefore proven that the dilepton production via virtual bremsstrahlung off unpolarized spin-one-half fermions is governed, in the leading and next-to-leading approximation, by the same formulas as that off pseudoscalar mesons.

Especially, if we neglect the terms proportional to derivatives of the unpolarized nonradiative matrix element squared, the tensor  $H^{\mu\nu}$  reduces to

$$H^{\mu\nu} = J^\mu J^\nu \overline{|\mathcal{M}_0|^2} \quad (3.13)$$

with  $J^\mu$  given by Eq. (2.8). The key leading term approximation relations (2.15), (2.18), and (2.19), which are of most practical interest, are valid also for virtual bremsstrahlung from unpolarized fermions.

It is a good check that our expression (2.35) for the virtual photon cross section meets, in the limit of zero photon mass, the unpolarized photon cross section calculated from the Burnett and Kroll [3] matrix element.

## IV. A SURVEY OF VIRTUAL BREMSSTRAHLUNG FORMULAS

### A. Rückl formula

In his work [7], Rückl suggested the formula

$$E_+ E_- \frac{d^6 \sigma^{e^+ e^-}}{d^3 p_+ d^3 p_-} = \frac{\alpha}{2\pi^2} \frac{1}{M^2} \left( \omega \frac{d^3 \sigma^\gamma}{d^3 q} \right)_{\mathbf{q}=\mathbf{p}_++\mathbf{p}_-}, \quad (4.1)$$

which links the cross section for production of dileptons via virtual photon bremsstrahlung to the bremsstrahlung cross section for real photons. The meaning of the symbols is as follows:  $\mathbf{p}_+$  and  $\mathbf{p}_-$  are the momenta of leptons,  $E_\pm = (\mathbf{p}_\pm^2 + \mu^2)^{1/2}$  are their energies,  $M^2 = (p_+ + p_-)^2$  is the dilepton mass squared,  $\mathbf{q}$  is the momentum of photon,  $\omega = |\mathbf{q}|$  is its energy, and  $\alpha$  is the fine-structure constant. Because of the vanishing photon mass, it is impossible to satisfy the relation  $\omega = E_+ + E_-$  simultaneously with  $\mathbf{q} = \mathbf{p}_+ + \mathbf{p}_-$ .

Let us investigate now how Eq. (4.1) copes with general principles we mentioned in the Introduction. The principle of relativity [17] requires that any meaningful formula must be relativistically covariant. This implies that if we view the bremsstrahlung process from another (primed) frame, we must find the same relation among the transformed quantities as we did in the original frame:

$$E'_+ E'_- \frac{d^6 \sigma^{e^+ e^-}}{d^3 p'_+ d^3 p'_-} = \frac{\alpha}{2\pi^2} \frac{1}{M^2} \left( \omega' \frac{d^3 \sigma^\gamma}{d^3 q'} \right)_{\mathbf{q}'=\mathbf{p}'_++\mathbf{p}'_-}. \quad (4.2)$$

We have assumed for simplicity that the velocities of colliding particles are collinear and performed a boost along the collision axis (otherwise  $\sigma$ 's also acquire primes). The left-hand

sides of Eqs. (4.1) and (4.2) are obviously equal. Relativistic covariance will thus be satisfied if and only if

$$\left( \omega \frac{d^3 \sigma^\gamma}{d^3 q} \right)_{\mathbf{q}=\mathbf{p}_++\mathbf{p}_-} = \left( \omega' \frac{d^3 \sigma^\gamma}{d^3 q'} \right)_{\mathbf{q}'=\mathbf{p}'_++\mathbf{p}'_-} . \quad (4.3)$$

But this condition cannot be fulfilled because in the new frame the photon momentum differs from the sum of lepton's momenta. In fact, for the longitudinal components of the corresponding vectors we have

$$\begin{aligned} p'_{\pm,L} &= \gamma (p_{\pm,L} - \beta E_{\pm}) \\ q'_L &= \gamma (q_L - \beta \omega) , \end{aligned} \quad (4.4)$$

which leads to

$$q'_L = p'_{+,L} + p'_{-,L} + \beta \gamma (E_+ + E_- - \omega) \quad (4.5)$$

with a nonvanishing extra term on the right-hand side.

In order to apply the global variable test, let us first use Eq. (2.17) to cast the Rückl formula in the form

$$E \frac{d^6 \sigma^{e^+e^-}}{dM^2 d^3 q d\Omega_+^*} = \frac{\alpha}{8\pi^2} \frac{1}{M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} \omega \frac{d^3 \sigma^\gamma}{d^3 q} . \quad (4.6)$$

Integration over the positron momentum angles is simple because nothing depends on them:

$$E \frac{d^4 \sigma^{e^+e^-}}{dM^2 d^3 q} = \frac{\alpha}{2\pi} \frac{1}{M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} \omega \frac{d^3 \sigma^\gamma}{d^3 q} . \quad (4.7)$$

This formula does not obviously have the form required by Eq. (1.1). If we nevertheless extract from it the virtual photon cross section, which is defined by (1.1), we arrive at

$$\left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{3\omega}{2\sqrt{\omega^2 + M^2}} \frac{M^2}{M^2 + 2\mu^2} \frac{d^3 \sigma^\gamma}{d^3 q} . \quad (4.8)$$

The  $M \rightarrow 0$  limit of this expression is zero. If we are not so strict and require only  $\mu \ll M \ll \omega$  (this is the situation met, e.g., in the low-mass, high-transverse-momentum dielectron production), we are left with another surprising relation

$$\left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{3}{2} \frac{d^3 \sigma^\gamma}{d^3 q} . \quad (4.9)$$

The presence of an incorrect numerical factor in the Rückl formalism was already noticed by Craigie [16], who ascribed it to the unjustified omission of the term  $l_\mu l_\nu$  in the lepton tensor [see Eq. (2.12)].

It is clear that the principal problems we have discussed here are not germane to the Rückl bremsstrahlung formalism, but are common to all the approaches where the dilepton cross section is assumed to be proportional to the photon cross section at the same momentum.

To obtain Eq. (4.1) in our formalism, we must (i) write  $p_i \cdot q = |\mathbf{q}| E_i - \mathbf{p}_i \cdot \mathbf{q}$  instead of the correct  $p_i \cdot q = (M^2 + \mathbf{q}^2)^{1/2} E_i - \mathbf{p}_i \cdot \mathbf{q}$  in Eq. (2.8); (ii) omit the term  $l_\mu l_\nu$  in Eq. (2.12); (iii) neglect  $q^\mu$  in the numerators and  $M^2$  in the denominators of Eq. (2.8). The first approximation is fatal for Lorentz covariance, the second one for the global variable test.

## B. Modifications of the Rückl formula

The Rückl formula (4.1) was often used in the calculation of dilepton yield in experimental and theoretical works. Several modifications of it have been put forward. As will be shown later, they brought improvement in a pragmatic sense, but were not able to cure its principal drawbacks.

Gale and Kapusta [10] replaced the photon energy squared  $q_0^2$  ( $\omega^2$  in our notation) in the denominator in their Eq. (5) by a symmetrized combination  $E(E^2 - M^2)^{1/2}$ , where  $E = E_+ + E_-$  is the dilepton energy. The quantity  $(E^2 - M^2)^{1/2}$  represents the dilepton momentum and as such must be equal to the photon momentum  $|\mathbf{q}|$ , which is in turn equal to the photon energy  $q_0$ . The replacement  $q_0^2 \rightarrow E(E^2 - M^2)^{1/2}$  is thus equivalent to multiplying the right-hand side of Eq. (4.1) by  $\omega/E$ . The same modification was used by Haglin, Gale, and Emel'yanov in [26] and also in a part of the paper [27] by Cleymans, Redlich, and Satz.

In a subsequent paper [11], Gale and Kapusta introduced a factor which partially corrected the soft photon approximation for processes with two particles in the final state. It accounts for the shrinking of the Lorentz invariant phase space available to them, which results from the emission of a dilepton with mass  $M$  and center-of-mass-system energy  $E^*$ . The production of dileptons that would violate the energy-momentum conservation is forbidden. The correction factor is given by (we display it in a simpler form assuming equal masses  $m$  for the final-state hadrons)

$$R(s, M, E^*) = \sqrt{\frac{s(s_2 - 4m^2)}{s_2(s - 4m^2)}} , \quad (4.10)$$

where  $s$  is the invariant energy available for all final-state particles and

$$s_2 = s + M^2 - 2E^*\sqrt{s} . \quad (4.11)$$

In a paper [14] of Haglin, Gale, and Emel'yanov, the photon energy squared  $q_0^2$  enters the denominator of the quantity  $|\epsilon \cdot J|_{ab \rightarrow cd}^2$  [see their (3.6), (3.7) or (A9)]. It is forced to acquire the value of  $(E_+ + E_-)^2$  by the second  $\delta$ -function in their Eq. (3.4). It induces a multiplicative factor of  $\omega^2/E^2$  on the right-hand side of the Rückl formula (4.1). The authors also used the correction factor (4.10). Recently, Haglin and Gale [28] utilized the same modification of the Rückl formula to assess the bremsstrahlung contribution to the  $e^+e^-$  invariant mass distribution in proton-proton and proton-neutron collisions at the lab kinetic energy of 4.9 GeV. For the final states with more than two hadrons they modified the phase-space correction factor accordingly.

## C. Paper by Craigie and Thompson

After a thorough discussion of the real photon bremsstrahlung, the authors of [8] turned to dileptons. If they had really done what they described verbally at the bottom of p. 129, they would have obtained immediately a simple and correct formula for the double differential cross section in the momenta of leptons [our Eq. (2.15)] with a little different Ansatz for the four-vector  $J$  (same as was used later in [12]). Unfortunately, they instead wrote a

cumbersome and obviously wrong formula (3.1) on p. 130. The incorrectness of the latter can be seen, e.g., from the fact that the quantity  $\text{Tr}\{\rho L\}$ , which enters it, depends on the momenta of hadrons via the four-vector  $J$  [see their Eq. (3.2)]. But there is no integration over hadron momenta in (3.1) [only that hidden in  $2q_0 d\sigma/d^3q$ , given by Eq. (2.5)]. The left-hand side of formula (3.1) in [8] thus depends on hadron momenta, which makes it unusable for evaluating the inclusive dilepton cross section.

The authors probably tried to express the double differential cross section as proportional to the cross section for virtual photon production. But, as we have learnt, this is possible only for the dilepton cross section in global dilepton variables [compare our (2.15) and (2.19)].

#### D. Formula used by Goshaw *et al.*

The experimentally observed production of very-low-energy  $e^+e^-$  pairs in 18 GeV/ $c$   $\pi^\pm p$  collisions was reported and compared to the expectations based on the leading-term bremsstrahlung calculations in the paper [9]. The authors used the formula

$$\begin{aligned} \frac{d\sigma}{d\mu d\omega d\Omega d\Omega^*} &= \frac{\alpha^2}{(2\pi)^4} \frac{[\mu^2 - (2m)^2]^{1/2} (\omega^2 - \mu^2)^{1/2}}{\mu^2 \omega^2} \\ &\times \int d^3P_3 \cdots d^3P_n \left[ \frac{(J \cdot l)^2}{\mu^2} - (J \cdot J) \right] \frac{d\sigma_h}{d^3P_3 d^3P_4 \cdots d^3P_n} , \end{aligned} \quad (4.12)$$

where  $\mu$ ,  $\omega$ , and  $d\Omega$  are the dilepton mass, energy, and infinitesimal solid angle, respectively,  $d\Omega^*$  is the infinitesimal solid angle of positron in the dilepton rest frame,  $l = P_+ - P_-$ , and

$$J = \omega \sum_{i=1}^n \frac{Q_i}{(P_+ + P_-) \cdot P_i} P_i . \quad (4.13)$$

The charge quantum number of outgoing particles ( $i = 3, \dots, n$ ) is  $Q_i$ , of incoming ( $i = 1, 2$ ) ones ( $-Q_i$ ). After changing the notation used in [9] to ours, Eq. (4.12) reads

$$E \frac{d^6\sigma^{e^+e^-}}{dM^2 d^3q d\tilde{\Omega}_+} = \frac{\alpha^2}{32\pi^4} \frac{1}{M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} \int \left[ \frac{1}{M^2} (l \cdot J_G)^2 - J_G^2 \right] d\sigma_0 , \quad (4.14)$$

where now

$$J_G^\mu = \sum Q'_i \frac{p_i^\mu}{p_i \cdot q} . \quad (4.15)$$

The formula (4.14) differs from our Eq. (2.18) by the sign of one of the terms in brackets. It is difficult to trace the origin of this discrepancy. The derivation of the formula (4.14) has not been published, although it was signalized in [9]. We suspect that Eq. (4.14) is not correct, because after integrating it over the positron momentum directions we get the formula

$$E \frac{d^4\sigma^{e^+e^-}}{dM^2 d^3q} = \frac{\alpha^2}{6\pi^3} \frac{1}{M^2} \left( 1 - \frac{\mu^2}{M^2} \right) \sqrt{1 - \frac{4\mu^2}{M^2}} \int (-J_G^2) d\sigma_0 , \quad (4.16)$$

which does not comply with the global variable test, defined by Eqs. (1.1) and (1.3). In this respect, the fact that the authors used additional approximations to get their Eq. (4.13) [compare (2.8) and (4.15)] seems to be of lesser importance.

### E. Formula of Balek, Pišútová, and Pišút

In paper [12] the formula analogous to our (2.19) was written for a charged particle scattering on a neutral particle (or on a potential). Their formalism satisfies both the Lorentz covariance and global variable tests. However, instead of the vector  $J^\mu$  given by Eq. (2.8), the following one was chosen:

$$C^\mu = \frac{p_1^\mu}{p_1 \cdot q} - \frac{p_a^\mu}{p_a \cdot q} . \quad (4.17)$$

Here,  $p_a$  and  $p_1$  are four-momenta of the charged particle before and after the scattering, respectively. In this case, Eq. (2.8) becomes

$$J^\mu = \frac{(2p_1 + q)^\mu}{2 p_1 \cdot q + M^2} - \frac{(2p_a - q)^\mu}{2 p_a \cdot q - M^2} . \quad (4.18)$$

To get the Ansatz (4.17) of Balek *et al.* we have to neglect  $q^\mu$  in the numerators and  $M^2$  in the denominators above.

For later convenience we write here the  $n$ -final-particle generalization of the Balek *et al.* formula in our notation. The same conventions as used in (2.23) apply.

$$E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{\alpha}{4\pi^2} \int - \left( \sum Q'_i \frac{p_i^\mu}{E_i E - \mathbf{p}_i \cdot \mathbf{q}} \right)^2 d\sigma_0 . \quad (4.19)$$

We will return to the approximation of Balek, Pišútová and Pišút in Appendix A, in connection with the concept of transverse-mass scaling (see below).

### F. Formula of Cleymans, Goloviznin, and Redlich

The authors of [13] used the following classical-electrodynamics motivated expression for the energy per unit of momentum radiated in the form of virtual photons with mass  $M$  if the charged particle changes its velocity from  $\mathbf{v}_1$  to  $\mathbf{v}_2$  (we switch from their notation to ours):

$$\frac{d^3 I}{d^3 q} = \frac{\alpha}{4\pi^2} \left| \frac{\mathbf{n} \times \mathbf{v}_2}{E - \mathbf{v}_2 \cdot \mathbf{q}} - \frac{\mathbf{n} \times \mathbf{v}_1}{E - \mathbf{v}_1 \cdot \mathbf{q}} \right|^2 . \quad (4.20)$$

Our expression for this quantity stems from the relations

$$\frac{d^3 I}{d^3 q} = E \frac{d^3 N^{\gamma^*}}{d^3 q} = \frac{1}{\sigma_0} E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M , \quad (4.21)$$

where  $N^{\gamma^*}$  is the mean number of virtual photons with mass  $M$  per a collision, in conjunction with Eqs. (2.21) and (2.20). Choosing the cross section  $d\sigma_0$  that allows only the required change of velocity, we get



$$\frac{d^3 I}{d^3 q} = \frac{\alpha}{4\pi^2} \left[ (\mathbf{n} \times \mathbf{J})^2 + \frac{M^2}{E^2} (\mathbf{n} \cdot \mathbf{J})^2 \right] \quad (4.22)$$

with

$$\mathbf{J} = \frac{\mathbf{v}_2 + \mathbf{q}/(2E_2)}{E - \mathbf{v}_2 \cdot \mathbf{q} + M^2/(2E_2)} - \frac{\mathbf{v}_1 - \mathbf{q}/(2E_1)}{E - \mathbf{v}_1 \cdot \mathbf{q} - M^2/(2E_1)}. \quad (4.23)$$

Comparing (4.20) with (4.22) and (4.23) we can see that in [13] two additional approximations have tacitly been made in contrast to our formalism: (i) The terms proportional to  $\mathbf{q}$  in the numerators and those proportional to  $M^2$  in the denominators of Eq. (4.23) have been neglected. This is equivalent to the approximation made in [12]. (ii) The second term in the brackets of (4.22) has not been considered. This approximation is more dangerous, since it makes the quantity (4.20) non-covariant under Lorentz transformations.

### G. Transverse mass scaling

In this and next subsections we are going to report about two attempts to relate the virtual photon cross section to the experimentally accessible cross section for real photon production.

Farrar and Frautschi [15] and Cobb *et al.* [29] used the concept of transverse mass scaling. They assumed that the virtual photon cross section does not depend on three variables (longitudinal momentum  $q_L$ , transverse momentum  $q_T$ , and virtual photon mass  $M$ ), but rather on only two [ $q_L$  and transverse mass  $M_T = (q_T^2 + M^2)^{1/2}$ ]. The condition (1.3) then leads to the relation

$$\left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{d^3 \sigma^\gamma}{d^3 p}, \quad (4.24)$$

where  $p_L = q_L$  and  $p_T = (q_T^2 + M^2)^{1/2}$ . Because both the energies and longitudinal momenta of real and virtual photons are now equal, the relation (4.24) is covariant under longitudinal Lorentz boosts. In our test process  $\rho^0 \rightarrow \pi^+ \pi^- e^+ e^-$ , this approach will be very successful.

The origin and limitations of the transverse-mass scaling from the point of view of the leading term bremsstrahlung formalism are discussed in Appendix A.

### H. Real photon approximation

Blockus *et al.* [30] conjectured that the formula

$$\left( E^* \frac{d^3 \sigma^{\gamma^*}}{d^3 q^*} \right)_M = \frac{f(M, q_L^*, q_T)}{f(0, q_L^*, q_T)} \left( \omega^* \frac{d^3 \sigma^\gamma}{d^3 q^*} \right) \quad (4.25)$$

is valid in the center-of-mass frame (we use  $q_L^*$  instead of their  $x = 2q_L^*/\sqrt{s}$ ). The quantity  $f$  is the structure function for the production of a virtual photon of mass  $M$  summed over photon polarization states. The authors considered two options for their ratio entering the

right-hand side of Eq. (4.25): (i) independent of  $M$  (i.e., identically equal to 1), and (ii) linearly dependent on  $M$ . None of these options seemed to be excluded by their data integrated over the region of acceptance. In our numerical comparison of various bremsstrahlung formulas we will explore the former option

$$\left(E^* \frac{d^3\sigma^{\gamma*}}{d^3q^*}\right)_M = \omega^* \frac{d^3\sigma^\gamma}{d^3q^*}, \quad (4.26)$$

and will refer to it as the “real photon approximation”.

## V. A MODEL OF TWO-PION RADIATIVE DECAYS OF $\rho^0$

As noted earlier, we will check the reliability of the various leading term virtual bremsstrahlung formulas by applying them to the strong-interaction decay  $\rho^0 \rightarrow \pi^+\pi^-$  in order to get estimates of the differential decay width for

$$\rho^0 \rightarrow \pi^+\pi^-e^+e^-. \quad (5.1)$$

While a comparison of the bremsstrahlung formulas themselves is instructive, additional insight may be gained by also comparing them to results from a formalism that goes beyond the leading term approximation. To our knowledge, nobody has investigated the decay (5.1) theoretically yet, probably because the chance to detect it experimentally is very meager. We present some estimates in Subsection B.

The real photon counterpart of (5.1), namely  $\rho^0 \rightarrow \pi^+\pi^-\gamma$  has been dealt with by several authors [21,31,32]. In this work we will adopt Singer’s approach [21], which agrees nicely with the experimental data (see below). We will first recapitulate its main points and then generalize it to the massive photon (dilepton) case.

To simplify notation, we introduce the following ratio of decay widths

$$B^\gamma = \frac{\Gamma_{\rho^0 \rightarrow \pi^+\pi^-\gamma}}{\Gamma_{\rho^0 \rightarrow \pi^+\pi^-}}. \quad (5.2)$$

We will call it a branching ratio in spite of a small incorrectness this introduces ( $\Gamma_{\rho^0 \rightarrow \pi^+\pi^-}$  is smaller than the total width of  $\rho^0$  by about 1 %). The quantities  $B^{\gamma*}$  and  $B^{e^+e^-}$  will have analogous meaning.

### A. Decay $\rho^0 \rightarrow \pi^+\pi^-\gamma$

Strong interaction dynamics enters Singer’s calculation through the assumption about the  $\rho^0\pi^+\pi^-$  vertex in the form

$$V^\alpha = f_{\rho\pi\pi}(p_1 - p_2)^\alpha, \quad (5.3)$$

where  $p_1$  ( $p_2$ ) is the four-momentum of  $\pi^+$  ( $\pi^-$ ). For external  $\rho$ ’s, this is to be contracted with the polarization vector  $\epsilon_\alpha(\lambda_\rho)$ . The coupling constant  $f_{\rho\pi\pi}$  can be fixed by utilizing the  $\rho \rightarrow \pi\pi$  decay width.

When electromagnetic interactions are switched on by the minimal interaction principle, the vertex (5.3) generates a contact term

$$C^{\mu\alpha} = -2ef_{\rho\pi\pi}g^{\mu\alpha}, \quad (5.4)$$

which must be considered in addition to the usual vertices of pseudoscalar electrodynamics (see Fig. 2). It should be contracted with both the  $\rho$  and  $\gamma$  lines which enter the contact vertex together with the two pion lines. Combining the resulting expressions for  $\rho^0 \rightarrow \pi^+\pi^-$  and  $\rho^0 \rightarrow \pi^+\pi^-\gamma$  decay widths from [21], we obtain the formula

$$\begin{aligned} \frac{dB^\gamma}{d\omega^*} = & \frac{4\alpha}{\pi} \frac{1}{(m_\rho^2 - 4m_\pi^2)^{3/2}} \frac{1}{\omega^*} \left\{ \omega_m^* (m_\rho^2 - 2m_\pi^2 - 2m_\rho\omega^*) \ln \frac{1+\xi}{1-\xi} \right. \\ & \left. - m_\rho \xi \left[ \omega_m^* (m_\rho - 2\omega^*) - 2\omega^{*2} \right] \right\}, \end{aligned} \quad (5.5)$$

where  $\omega^*$  is the photon energy in the  $\rho^0$  rest frame and

$$\xi = \sqrt{\frac{2(\omega_m^* - \omega^*)}{m_\rho - 2\omega^*}}. \quad (5.6)$$

The maximum value of  $\omega^*$  is  $\omega_m^* = (m_\rho^2 - 4m_\pi^2)/(2m_\rho)$ .

After integrating Eq. (5.5) over photon energies greater than 50 MeV we get the branching ratio of 1.12 %, in nice agreement with the experimental value  $(0.99 \pm 0.16)$  % [33]. Also, the distribution in photon energies (depicted in Fig. 3 by solid curve) agrees remarkably well with experiment [22].

## B. Decay $\rho^0 \rightarrow \pi^+\pi^-e^+e^-$

As has already been stressed, in order to know the differential decay width in global dilepton variables of the decay (5.1), it is sufficient to calculate the differential decay width for  $\rho^0 \rightarrow \pi^+\pi^-\gamma^*$  with a massive photon. As was proven in [18], the evaluation of the latter is governed by the same Feynman rules as in the case of a real photon. What changes are the kinematical relations which must accommodate the non-vanishing photon mass. As a result, we obtain

$$\begin{aligned} \left( \frac{dB^{\gamma^*}}{dq^*} \right)_M = & \frac{4\alpha}{\pi} \frac{1}{(m_\rho^2 - 4m_\pi^2)^{3/2}} \frac{q^*}{E^{*2}} \\ & \times \left\{ \left[ E_m^* (m_\rho^2 - 2m_\pi^2 - 2E_m^*m_\rho) + M^2 \left( E_m^* + \frac{m_\rho}{4} \right) \right] \ln \frac{E^* + \xi q^*}{E^* - \xi q^*} \right. \\ & \left. - E^* q^* \xi \left[ \frac{(m_\rho - 2E_m^*)(2m_\rho E_m^* - M^2)}{2(E^{*2} - \xi^2 q^{*2})} - 2m_\rho \right] \right\}. \end{aligned} \quad (5.7)$$

Here,

$$E_m^* = \frac{m_\rho^2 + M^2 - 4m_\pi^2}{2m_\rho} \quad (5.8)$$

is the maximum value of  $E^*$ , the massive photon energy in the  $\rho^0$ -rest frame and

$$\xi = \sqrt{\frac{2m_\rho (E_m^* - E^*)}{m_\rho^2 + M^2 - 2m_\rho E^*}}. \quad (5.9)$$

The interested reader may find a few intermediate steps in Appendix B. Integrating the differential branching ratio for  $\rho^0 \rightarrow \pi^+ \pi^- e^+ e^-$

$$\frac{d^2 B^{e^+ e^-}}{dM^2 dq^*} = \mathcal{T}(M^2) \left( \frac{dB^{\gamma^*}}{dq^*} \right)_M \quad (5.10)$$

over the dilepton momenta and masses, we get the value  $1.5 \times 10^{-4}$  for  $\rho^0 \rightarrow \pi^+ \pi^- e^+ e^-$  and  $4.9 \times 10^{-7}$  for  $\rho^0 \rightarrow \pi^+ \pi^- \mu^+ \mu^-$ . A low-dielectron-mass cut of  $50 \text{ MeV}/c^2$  ( $100 \text{ MeV}/c^2$ ) reduces the branching ratio to  $1.1 \times 10^{-5}$  ( $4.0 \times 10^{-6}$ ).

## VI. BREMSSTRAHLUNG FORMULAS FOR TWO-PION RADIATIVE DECAYS OF $\rho^0$

Our goal here is to calculate the differential branching ratio of  $\rho^0 \rightarrow \pi^+ \pi^- e^+ e^-$  within all approaches to dilepton bremsstrahlung we found in the literature. Because some approaches [12,13] do not provide the distribution in the momenta of electrons, we will concentrate on a less general distribution in global dilepton variables. For later use and reference we first explore the simpler case of the  $\rho^0$  decay to a dipion and photon.

### A. Decay $\rho^0 \rightarrow \pi^+ \pi^- \gamma$

In order to calculate the branching ratio in the leading term bremsstrahlung approximation, we start with the formula

$$\frac{dB^\gamma}{d\omega^*} = \frac{\alpha\omega^*}{4\pi^2} \int (-J_R^2) d\Omega_{\mathbf{q}^*}, \quad (6.1)$$

which is a real photon mutation of Eq. (2.42). Using Eqs. (2.21) and (2.23), we can write the integrand in the form

$$-J_R^2 = \left( \frac{2v^*}{\omega^*} \right)^2 \frac{\sin^2 \alpha^*}{(1 - v^{*2} \cos^2 \alpha^*)^2}, \quad (6.2)$$

where

$$v^* = \sqrt{1 - \frac{4m_\pi^2}{m_\rho^2}} \quad (6.3)$$

is the speed of pions and  $\alpha^*$  the angle between the photon and  $\pi^+$  momenta in the  $\rho^0$  rest frame. An elementary integration gives us

$$\frac{dB^\gamma}{d\omega^*} = \frac{2\alpha}{\pi\omega^*} \left( \frac{1+v^{*2}}{2v^*} \ln \frac{1+v^*}{1-v^*} - 1 \right). \quad (6.4)$$

A numerical evaluation shows (Fig. 3, dashed curve) that the leading term bremsstrahlung formula (6.4) exceeds the data, especially at large momenta. This trend is understandable, as the formula ignores the energy-momentum-conservation constraints. The branching ratio for  $\omega^* > 50$  MeV is 2.05 %, almost twice as much as the experimental observation. For later reference, we also present in Fig. 4 the ratio of the leading-term-bremsstrahlung branching ratio (6.4) to that calculated from Singer's model (5.5) as a function of the photon momentum  $q^*$  (which is, of course, equal to its energy  $\omega^*$ ).

### B. Decay $\rho^0 \rightarrow \pi^+\pi^-e^+e^-$

Our aim here is to derive formulas for the differential branching ratios in dilepton momentum and mass  $d^2B^{e^+e^-}/dM^2dq^*$  in the various formalisms. We will apply the same assumptions and approximations which the various authors have used when deriving their formulas for the differential cross section. For the formalisms that satisfy the global variable test ([12,13], our formalism), it is sufficient to calculate the branching ratio for the virtual photon production

$$\left( \frac{dB^{\gamma^*}}{dq^*} \right)_M = \frac{\alpha}{4\pi^2} \frac{q^{*2}}{E^*} \int (-J^2) d\Omega_{\mathbf{q}^*} \quad (6.5)$$

[cf. (2.42)] and then multiply it by the function  $\mathcal{T}(M^2)$ , given by Eq. (1.2). In other cases, the procedure will be less straightforward.

Applying our formula (2.8) to the process considered here, we can write

$$\mathbf{J} = \frac{2m_\rho^{-1}}{(E^* + M^2/m_\rho)^2 - (v^*q^* \cos \theta^*)^2} \left[ (E^*m_\rho + M^2)\mathbf{v}^* + v^*q^* \cos \theta^* \mathbf{q}^* \right]. \quad (6.6)$$

Now we insert this into (2.21) and calculate the integral which enters the formula (6.5). We finish with

$$\begin{aligned} \frac{d^2B^{e^+e^-}}{dM^2dq^*} = & \frac{2\alpha}{\pi} \frac{q^*}{E^*} \mathcal{T}(M^2) \left[ \frac{1+v^{*2} - M^2/m_\rho^2}{2v^*(E^* + M^2/m_\rho)} \ln \frac{E^* + M^2/m_\rho + v^*q^*}{E^* + M^2/m_\rho - v^*q^*} \right. \\ & \left. - \frac{(1-v^{*2} - M^2/m_\rho^2)q^*}{E^{*2} - (v^*q^*)^2 + (2E^* + M^2/m_\rho)M^2/m_\rho} \right]. \end{aligned} \quad (6.7)$$

This is the leading term bremsstrahlung formula based on the formalism we developed in Sec. II. The ratio of (6.7) to the “exact” formula (5.10) is presented as a function of dielectron momentum  $q^*$  for several dielectron masses in Figs. 5 through 9 by a solid curve.

The corresponding formula in the Balek, Pišútová, and Pišút [12] approximation is obtained along the same lines. The only difference is in using their (4.17), which, translated from the scattering case to the  $\rho^0 \rightarrow \pi^+\pi^-$  decay, reads

$$\mathbf{J} = \frac{2E^*}{E^{*2} - (v^*q^* \cos \theta^*)^2} \mathbf{v}^*, \quad (6.8)$$

instead of our (6.6). The result is

$$\frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{2\alpha}{\pi} \frac{q^*}{E^{*2}} \mathcal{T}(M^2) \left[ \frac{1+v^{*2}}{2v^*} \ln \frac{E^* + v^*q^*}{E^* - v^*q^*} - \frac{(1-v^{*2}) E^* q^*}{E^{*2} - (q^*v^*)^2} \right]. \quad (6.9)$$

It is a good check that for  $M \ll m_\rho$ , Eq. (6.7) tends to agree with (6.9).

When we combined the  $M_T$ -scaling hypothesis (4.24) with the leading term bremsstrahlung formula for real photons (2.22), we got the same result as from the Balek *et al.* approximation (6.9). This prompted us to investigate the connection between the two approaches in more detail in Appendix A.

In order to calculate the differential branching ratio in the Cleymans, Goloviznin, and Redlich approximation [13], we also use (6.8). In addition, we have to discard the dot product in Eq. (2.21). In this case we get

$$\frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{2\alpha}{\pi} \frac{1}{E^*} \mathcal{T}(M^2) \left[ \frac{E^{*2} + (q^*v^*)^2}{2v^* E^* q^*} \ln \frac{E^* + v^*q^*}{E^* - v^*q^*} - 1 \right]. \quad (6.10)$$

To find the branching ratios in remaining approximations no additional calculations are needed; we need only to combine the proper formulas.

The Rückl approximation formula (4.7) in terms of branching ratios may be written as

$$E^* \frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{\alpha}{2\pi} \frac{1}{M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} q^* \frac{dB^\gamma}{dq^*}. \quad (6.11)$$

Merging this with (6.4), we get

$$\frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{\alpha^2}{\pi^2} \frac{1}{E^* M^2} \sqrt{1 - \frac{4\mu^2}{M^2}} \left( \frac{1+v^{*2}}{2v^*} \ln \frac{1+v^*}{1-v^*} - 1 \right). \quad (6.12)$$

The Gale and Kapusta [10] modification of the Rückl formula consists in multiplying (6.12) by the factor  $q^*/E^*$ . To account for the later modification by the same authors [11], we must also include their phase-space correction factor  $R(m_\rho^2, M, E^*)$ , see (4.10).

To adopt the Haglin, Gale and Emely'anov [14] modification of the Rückl formula, we have to multiply (6.12) by  $R(m_\rho^2, M, E^*)(q^*/E^*)^2$ .

When dealing with the Goshaw *et al.* [9] formalism, we first rewrite Eq. (4.16) by means of (6.5) to the form

$$\frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{2\alpha}{3\pi} \frac{1}{M^2} \left( 1 - \frac{\mu^2}{M^2} \right) \sqrt{1 - \frac{4\mu^2}{M^2}} \left( \frac{dB^{\gamma^*}}{dq^*} \right)_M. \quad (6.13)$$

The virtual-photon branching ratio on the right-hand side can be taken from the Balek *et al.* formalism, because they use the identical four-vector  $J$  [compare (4.15) with (4.17)]. It can easily be read off the Eq. (6.9); one simply ignores the function  $\mathcal{T}(M^2)$ . As a result, we get

$$\begin{aligned} \frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = & \frac{4\alpha^2}{3\pi^2} \frac{q^*}{(E^* M)^2} \left(1 - \frac{\mu^2}{M^2}\right) \sqrt{1 - \frac{4\mu^2}{M^2}} \left[ \frac{1 + v^{*2}}{2v^*} \ln \frac{E^* + v^* q^*}{E^* - v^* q^*} \right. \\ & \left. - \frac{(1 - v^{*2}) E^* q^*}{E^{*2} - (q^* v^*)^2} \right]. \end{aligned} \quad (6.14)$$

The real photon approximation (4.26) combined with (5.10) and (6.4) leads to

$$\frac{d^2 B^{e^+e^-}}{dM^2 dq^*} = \frac{2\alpha}{\pi E^*} \mathcal{T}(M^2) \sqrt{1 - \frac{4\mu^2}{M^2}} \left( \frac{1 + v^{*2}}{2v^*} \ln \frac{1 + v^*}{1 - v^*} - 1 \right). \quad (6.15)$$

All the formulas we have derived here are normalized to the “exact” formula [Eqs. (5.7) and (5.10)] and visualized as functions of dielectron momentum at fixed dielectron masses in Figs. 5 through 9.

## VII. COMMENTS AND CONCLUSIONS

We based our formalism for dilepton production via virtual photon bremsstrahlung on the following assumptions:

1. Only the Feynman diagrams in which a virtual photon line is attached to one of the external legs are important.
2. The charged particles that participate in the process are pseudoscalar mesons or unpolarized spin-one-half fermions.
3. The dilepton four-momentum in the argument of the Dirac  $\delta$ -function in Eq. (2.14) (or in similar relations for decays or processes with more than two particles in the initial state) may be neglected.
4. The modifications of electromagnetic interactions of hadrons induced by form factors and anomalous magnetic moments (in the case of fermions) are negligible.

To obtain the leading term approximation, we further neglected terms that are proportional to the derivatives of the nonradiative matrix element squared. Our notion of the leading term approximation is thus based on the proportionality of the bremsstrahlung cross section to that of the nonradiative reaction rather than on the order in dilepton momentum  $q$ . In our leading term formulas, we use the four-vector  $J$  that contains also subleading terms in  $q$ . The numerical comparison with the “exact” formula shows that it is beneficial.

In Sec. IV, we have pointed out the additional assumptions which are required to reduce our formulas to the virtual bremsstrahlung formulas that were derived or suggested previously. Of these formulas, only one, namely that of Balek, Pišútová and Pišút [12], satisfied both the Lorentz covariance and global variable tests.

The numerical analysis presented in Sec. VI shows that all leading term virtual bremsstrahlung formulas (including ours) overestimate the dilepton production, albeit to different degrees. As a matter of fact, also the real photon production is overestimated by

the standard leading-term formula (see Figs. 3 and 4). But due to the more complex nature of virtual bremsstrahlung, the situation with dileptons is more complicated. Besides the excess in the high-momentum region, which is similar to that for real photons, some formulas also overshoot the yield at small momenta. These formulas relate dilepton production to the real photon cross section rather than to the virtual one. Near threshold, the differential cross section in virtual photon momentum behaves like  $q^{*2}$  [in both the “exact” formula (5.7) and Eq. (6.7)], whereas for real photons the behavior is  $q^{*-1}$  [see Eqs. (5.5) and (6.4)].

Energy-momentum conservation is not built into the leading term bremsstrahlung formulas. The fact that they overestimate the dilepton yield at high dilepton momenta is therefore not surprising. But we will see that it is only a part of the story.

Gale and Kapusta [11] introduced the factor (4.10), which accounts for the shrinking of the final-state phase space and prevents violation of energy-momentum conservation. They, and also the authors of a later work [14], combined it with the Rückl formula. This procedure obscures the conclusions somewhat, because the Rückl formula contains a wrong numerical factor. We therefore apply the correction factor (4.10) to the leading term virtual bremsstrahlung formula (6.7). The results are displayed in Fig. 10 after being normalized to the “exact” formula. Comparison with corresponding non-corrected curves shows that only a part of the excess over the “exact” formula has been removed. The remaining excess should be ascribed to other sources. It can be a modification of the nonradiative matrix element combined with a destructive interference between radiation from the external particles and from the contact term. Both these effects are neglected in the leading-term approximation.

The bremsstrahlung calculations play an important role in assessing the conventional sources of photons or dileptons in experiments aimed at revealing anomalous production of electromagnetic probes as a sign of new physics phenomena. For example, Haglin and Gale [28] have recently shown that bremsstrahlung is the largest source of low-mass dielectrons in 4.9 GeV  $pp$  collisions. A quite good, even if not perfect, agreement with experimental data [34] was achieved. The authors used the modified Rückl formula, discussed in IV B. But our toy example showed that this formula overestimates the dielectron yield by a factor of 2–3. This suggests that calculations using a more correct bremsstrahlung formalism have to be performed before a definite conclusion about the physics behind the Dilepton Spectrometer collaboration data [34] is drawn.

To conclude, we have derived formulas for differential cross sections of dilepton production via virtual photon bremsstrahlung from pseudoscalar mesons and unpolarized fermions in the leading and subleading approximations. These formulas satisfy the conditions imposed by Lorentz covariance and gauge invariance. From the practical point of view, the leading term formulas (2.15), (2.18), and (2.19) are the most important. They enable us to estimate the radiative cross sections on the basis of the corresponding nonradiative cross section. Comparing to previously published formulas, our formalism exhibits the best agreement with the exact formula when applied to a concrete physical process.

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## APPENDIX A: $M_T$ -SCALING AND VIRTUAL BREMSSTRAHLUNG FORMALISM

Let us describe the multiparticle production in a frame where the velocities of the two incident particles are collinear and define a collision axis. The dot product between the dilepton momentum  $\mathbf{q}$  and the momentum  $\mathbf{p}_i$  of the  $i$ th particle can then be written as

$$\mathbf{q} \cdot \mathbf{p}_i = |\mathbf{q}| p_{i,L} \cos \theta + \mathbf{q}_T \cdot \mathbf{p}_{i,T} , \quad (\text{A1})$$

where  $\theta$  is the angle between the dilepton momentum and the projectile velocity.

If the transverse momenta of the outgoing charged particles are negligibly small (the criterion of negligibility depends, of course, on the value of  $\theta$ ), the rightmost term in (A1) can be omitted. As a consequence, the virtual photon bremsstrahlung formula in the Balek *et al.* approximation (4.19) can be rearranged into

$$E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{\alpha}{4\pi^2 E^2} \int - \left( \sum \frac{Q'_i}{E_i} \frac{p_i^\mu}{1 - E^{-1} v_{i,L} |\mathbf{q}| \cos \theta} \right)^2 d\sigma_0 \quad (\text{A2})$$

with  $v_{i,L} = p_{i,L}/E_i$ . Let us introduce a new vector  $\mathbf{p}$  by

$$\begin{aligned} p_L &= q_L \\ \mathbf{p}_T &= \frac{\mathbf{q}_T}{q_T} \sqrt{q_T^2 + M^2} \end{aligned} \quad (\text{A3})$$

and denote its polar angle by  $\alpha$ . We have

$$\begin{aligned} \mathbf{p}^2 &= q_L^2 + q_T^2 + M^2 = E^2 , \\ \cos \alpha &= \frac{p_L}{|\mathbf{p}|} = \frac{|\mathbf{q}|}{E} \cos \theta . \end{aligned} \quad (\text{A4})$$

Equation (A2) then reads

$$E \left( \frac{d^3 \sigma^{\gamma^*}}{d^3 q} \right)_M = \frac{\alpha}{4\pi^2 \mathbf{p}^2} \int - \left( \sum \frac{Q'_i}{E_i} \frac{p_i^\mu}{1 - v_{i,L} \cos \alpha} \right)^2 d\sigma_0 . \quad (\text{A5})$$

The right-hand side is nothing else but the invariant cross section (2.22) for producing a real photon with the momentum  $\mathbf{p}$  and energy  $\omega = |\mathbf{p}|$  in the same approximation (negligible transverse momenta of hadrons). We have thus recovered the  $M_T$ -scaling hypothesis (4.24).

The  $M_T$ -scaling is clearly an approximate phenomenon. To arrive at it, we first made the approximations recapitulated in Sec. VII, that led to our basic formulas of Sec. II. Then we added another approximation to get the Balek *et al.* formula, which is for negligible transverse momenta equivalent to merging  $M_T$ -scaling with leading term formula for bremsstrahlung of real photons. In realistic situations, however, the violation coming from nonvanishing transverse momenta of hadrons also plays an important role. In our simple example—the nonradiative decay  $\rho^0 \rightarrow \pi^+\pi^-$  discussed in Sec. VI, the outgoing hadrons do not have any transverse momenta and the two approaches are fully equivalent.

## APPENDIX B: “EXACT” FORMULA FOR THE $\rho^0 \rightarrow \pi^+\pi^-\gamma^*$ BRANCHING RATIO

The matrix element is given by the Feynman diagrams depicted in Fig. 2 with a real photon replaced by a massive one. We define the invariant variables

$$\begin{aligned} s &= (p_1 + p_2)^2, \\ t' &= (p_2 + q)^2 - m_\pi^2, \\ u' &= (p_1 + q)^2 - m_\pi^2, \end{aligned} \quad (\text{B1})$$

with  $p_1$ ,  $p_2$ , and  $q$  being the four-momenta of  $\pi^+$ ,  $\pi^-$ , and  $\gamma^*$ , respectively. The invariants (B1) satisfy the usual relation

$$s + t' + u' = m_\rho^2 + M^2. \quad (\text{B2})$$

The sum over the  $\rho^0$  and  $\gamma^*$  polarizations of the matrix element squared is equal to

$$\sum_{\lambda_\rho, \lambda_{\gamma^*}} |\mathcal{M}|^2 = \left( \frac{ef_{\rho\pi\pi}}{t'u'} \right)^2 A^{\mu\alpha} A^{\nu\beta} \sum_{\lambda_\rho} \epsilon_\alpha(\lambda_\rho) \epsilon_\beta^*(\lambda_\rho) \sum_{\lambda_{\gamma^*}} \epsilon_\mu(\lambda_{\gamma^*}) \epsilon_\nu^*(\lambda_{\gamma^*}), \quad (\text{B3})$$

with the tensor  $A$  given by

$$A^{\mu\alpha} = t'(2p_1 + q)^\mu (p_1 - p_2 + q)^\alpha + u'(2p_2 + q)^\mu (p_2 - p_1 + q)^\alpha - 2t'u'g^{\mu\alpha}. \quad (\text{B4})$$

Thanks to the relations  $q_\mu A^{\mu\alpha} = 0$  and  $A^{\mu\alpha} P_\alpha = 0$ , where  $P$  is the  $\rho^0$  four-momentum, the sums of products of polarizations vectors in (B3) can be replaced by the corresponding metric tensors. After a little algebra, we get

$$\sum_{\lambda_\rho, \lambda_{\gamma^*}} |\mathcal{M}|^2 = 16\pi\alpha f_{\rho\pi\pi}^2 \left[ 2 + a_1 \left( \frac{1}{t'} + \frac{1}{u'} \right) - a_2 \left( \frac{1}{t'^2} + \frac{1}{u'^2} \right) \right], \quad (\text{B5})$$

where

$$\begin{aligned} a_1 &= \frac{1}{E_*^*} \left[ E_m^* (m_\rho^2 - 2m_\pi^2 - 2m_\rho E_*^*) + M^2 (E_m^* + m_\rho/4) \right], \\ a_2 &= \frac{1}{4} (m_\rho^2 - 4m_\pi^2) (4m_\pi^2 - M^2). \end{aligned} \quad (\text{B6})$$

See also (5.8). To calculate the invariant decay width of the unpolarized  $\rho^0$ , namely

$$E \left( \frac{d^3\Gamma_{\rho^0 \rightarrow \pi^+\pi^-\gamma^*}}{d^3q} \right)_M = \frac{\pi}{6m_\rho} \int \delta(P - q - p_1 - p_2) \sum_{\lambda_\rho, \lambda_{\gamma^*}} |\mathcal{M}|^2 \prod_{i=1}^2 \frac{d^3p_i}{2E_i(2\pi)^3} \quad (\text{B7})$$

we need integrals of the type

$$I_n = \int \frac{d^3p_1}{E_1} \int \frac{d^3p_2}{E_2} \delta(P - q - p_1 - p_2) \frac{1}{t'^n} , \quad (\text{B8})$$

which can be computed most simply in the two-pion rest frame after the substitutions  $Q = p_1 + p_2$  and  $R = p_1 - p_2$ . We get

$$\begin{aligned} I_0 &= 2\pi\xi , \\ I_1 &= \frac{\pi}{m_\rho q^*} \ln \frac{E^* + \xi q^*}{E^* - \xi q^*} , \\ I_2 &= \frac{2\pi\xi}{m_\rho^2} \frac{1}{E^{*2} - (\xi q^*)^2} , \end{aligned} \quad (\text{B9})$$

with  $\xi$  given by Eq. (5.9). Putting it all together and using the formula [21]

$$\Gamma_{\rho \rightarrow \pi\pi} = \frac{f_{\rho\pi\pi}^2}{48\pi m_\rho^2} (m_\rho^2 - 4m_\pi^2)^{3/2} , \quad (\text{B10})$$

we eventually get (5.7).

### APPENDIX C: LEADING TERM BREMSSTRAHLUNG AND EVENT GENERATORS

The attitude to the bremsstrahlung in hadronic reactions has been much influenced by the fact that the leading term in quantum electrodynamics is equivalent to the corresponding classical expression [24]. Similar relation exists also for virtual bremsstrahlung [12].

As a consequence, also the event generators constructed so far were, according to our knowledge, “classical.” The configuration of hadrons in momentum space was considered a source of photons or dileptons. The momenta of hadrons were assumed untouched by the creation of a photon or dilepton. This led to problems with the energy-momentum conservation.

We think that those problems can be cured by a more quantal approach. One should consider the transition probability from the initial state to the “complete” final state, containing both hadrons and a photon (or a dilepton). The energy-momentum conservation is firmly enforced. As we show below, the leading term bremsstrahlung approximation means that the probability of such a transition can be expressed as a product of two terms. The first of them is assumed to be independent of photon or dilepton four-momentum, the second one contains the momenta of both hadrons and a photon (a dilepton), but in a simple way. What is approximate is the transition probability (rephrased into the cross section, decay width, or other observable quantities), not the energy-momentum conservation.

To illustrate our point in more detail, let us return back to the basic equations. To simplify the discussion, we will consider the reaction

$$a + b \rightarrow 1 + 2 + \cdots + n + \gamma . \quad (\text{C1})$$

Its cross section is given by (we choose the center-of-mass reference frame)

$$\sigma = \frac{1}{4p_a^* \sqrt{s}} \int \sum_{\lambda_\gamma} |\mathcal{M}|^2 (2\pi)^4 \delta(p_a + p_b - \sum_i p_i - q) \frac{d^3 q}{2\omega(2\pi)^3} \prod_{i=1}^n \frac{d^3 p_i}{2E_i(2\pi)^3} . \quad (\text{C2})$$

Due to the presence of the four-dimensional  $\delta$ -function, only the  $(3n-1)$  momentum components are independent. It is convenient to transform the integration region in (C2) into a  $(3n-1)$ -dimensional unit cube. Several such procedures exist in the literature (see, e.g., [35–38]). We thus get

$$\sigma = \int \frac{d^{3n-1} \sigma}{d^{3n-1} \xi} d^{3n-1} \xi \quad (\text{C3})$$

with

$$\frac{d^{3n-1} \sigma}{d^{3n-1} \xi} = \frac{1}{4p_a^* \sqrt{s}} \sum_{\lambda_\gamma} |\mathcal{M}|^2 f(\xi_1, \xi_2, \cdots, \xi_{3n-1}) . \quad (\text{C4})$$

The function  $f$  results from the integration over the four dependent variables and from the Jacobian of the substitutions

$$\mathbf{p}_i = \mathbf{p}_i(\xi_1, \xi_2, \cdots, \xi_{3n-1}), \quad i = 1, 2, \cdots, n , \quad (\text{C5})$$

$$\mathbf{q} = \mathbf{q}(\xi_1, \xi_2, \cdots, \xi_{3n-1}) . \quad (\text{C6})$$

The momenta given by Eqs. (C5) and (C6) satisfy the energy-momentum conservation in reaction (C1). The above substitutions are not straightforward, they usually require to solve an algebraic equation. We refer the reader to the original literature [35–38].

The master equation of the event generator is equivalent to the evaluating of the cross section by a Monte Carlo method:

$$\sigma = \frac{1}{N} \sum_{k=1}^N \left( \frac{d^{3n-1} \sigma}{d^{3n-1} \xi} \right)_{\tilde{\xi}^{(k)}} , \quad (\text{C7})$$

where the sums runs over the random points uniformly distributed within the  $(3n-1)$ -dimensional unit cube. Each such point generates an “event”—a set of momenta of particles in the final state. The quantity (C4) is a weight that is assigned to each “event”. If we succeeded in finding such a form of substitutions (C5) and (C6) that the weights of all events are same, we would have an ideal event generator.

We have not made any approximation yet, Eqs. (C3) and (C4) are exact. Now, we are again going to consider only the radiation coming from the external particles and to ignore the possible changes in the strong “core” of the Feynman diagram if one of the external legs

goes off-shell. The sum of the matrix element squared over the photon polarizations is then simply equal to

$$\sum_{\lambda_\gamma} |\mathcal{M}|^2 = 4\pi\alpha |\mathcal{M}_0|^2 \left(-J_R^2\right) , \quad (\text{C8})$$

where  $J_R$  is given by Eq. (2.23). To relate  $|\mathcal{M}_0|^2$  to observable quantities, we need a set of hadron momenta that satisfy the energy-momentum conservation for the nonradiative reaction (2.1). In other words, for the sake of evaluation of the integrand in (C4) we must “spoil” the momenta  $\mathbf{p}_i$  a little. This procedure replaces the assumption that the four-vector  $q$  in the argument of the  $\delta$ -function in (2.14) may be neglected. We can change the hadron momenta in many different ways. Here is one of them.

We first express  $\xi_{3n-3}$ ,  $\xi_{3n-2}$ , and  $\xi_{3n-1}$  in terms of  $\mathbf{q}$  and remaining  $\xi$ ’s by inverting Eq. (C6). The momenta of hadrons thus become functions of  $\xi_1, \xi_2, \dots, \xi_{3n-4}$ , and  $\mathbf{q}$ . Now we can define a set of hadron momenta

$$\mathbf{p}_i^{(0)} = \lim_{\mathbf{q} \rightarrow \mathbf{0}} \mathbf{p}_i(\xi_1, \xi_2, \dots, \xi_{3n-4}, \mathbf{q}) , \quad (\text{C9})$$

which satisfy the energy-momentum conservation for reaction (2.1). We can use them to evaluate (C8) by means of the purely hadronic matrix element, which is related to the cross section of reaction (2.1). The weight of the  $k$ th event thus becomes

$$w_k = \frac{4\pi\alpha}{4p_a^* \sqrt{s}} |\mathcal{M}_0|^2_{\left\{\mathbf{p}_i^{(0)}\right\}_k} \left(-J_R^2\right)_{\{\mathbf{p}_i, \mathbf{q}\}_k} f\left(\xi_1^{(k)}, \dots, \xi_{3n-1}^{(k)}\right) . \quad (\text{C10})$$

The above procedure can be used if the exclusive cross section is given by an analytic formula.

In a more realistic situation, the bremsstrahlung cross section is estimated from the experimental data on the nonradiative reaction (2.1) on event-by-event basis. The Nature acts as an “ideal event generator”, which generates exclusive sets of hadron momenta  $\mathbf{p}_i^{(0)}$ ,  $i = 1, \dots, n$ . Each set is assigned the same weight  $\sigma_0$ . The distribution of events in hadron momenta is governed by the matrix element squared  $|\mathcal{M}_0|^2$  and the phase space. The role of the latter can again be described using the substitutions

$$\mathbf{p}_i^{(0)} = \mathbf{p}_i^{(0)}(\xi_1, \xi_2, \dots, \xi_{3n-4}) , \quad i = 1, 2, \dots, n \quad (\text{C11})$$

that guarantee the energy-momentum conservation in reaction (2.1). For properly chosen substitutions (C11), the  $\xi$ -dependence of the partly integrated Jacobian  $f^{(0)}$  compensates the  $\mathbf{p}$ -dependence of the matrix element squared in the sense that the product of them is constant. The weight of each event thus is, as required,

$$\sigma_0 = \frac{1}{4p_a^* \sqrt{s}} |\mathcal{M}_0|^2 f^{(0)}(\xi_1, \dots, \xi_{3n-4}) . \quad (\text{C12})$$

The momentum distribution of experimental events corresponds to a uniform distribution of points within the  $(3n-4)$ -dimensional unit cube. Their coordinates are given by the inverse substitution

$$\xi_i = \xi_i(\mathbf{p}_1^{(0)}, \mathbf{p}_2^{(0)}, \dots, \mathbf{p}_n^{(0)}) , \quad i = 1, \dots, 3n-4 . \quad (\text{C13})$$

To generate a photon, we proceed in two steps. Firstly, for each event, we have to “spoil” the hadron momenta and add a momentum of photon in such a way that they together cope with the energy-momentum constraints for the reaction (C1). Secondly, we have to find the weight of this photonic event.

In order to obtain the momenta in the radiative event, we supplement the coordinates (C13) by three random numbers  $\xi_{3n-3}$ ,  $\xi_{3n-2}$ , and  $\xi_{3n-1}$ . The “spoiled” hadron momenta and the photon momentum are now given by Eqs. (C5) and (C6). The weight of an event can be found by inserting  $|\mathcal{M}_0|^2$  from Eq. (C12) to Eq. (C10).

$$w'_k = 4\pi\alpha \sigma_0 \left(-J_R^2\right)_{\{\mathbf{p}_i, \mathbf{q}\}_k} \frac{f\left(\xi_1^{(k)}, \dots, \xi_{3n-1}^{(k)}\right)}{f^{(0)}\left(\xi_1^{(k)}, \dots, \xi_{3n-4}^{(k)}\right)}. \quad (\text{C14})$$

Of course, the substitutions (C5) and (C11) cannot be independent. They must satisfy the condition (C9). In actual calculation, we do not usually know the ideal substitution (C11) for momenta in the nonradiative reaction, and work with a substitution that gives fluctuating right-hand side of Eq. (C12). The weights of the photon-producing events (C14) should be influenced only little, because they contain the ratio  $f/f^{(0)}$ .

The dilepton generators in either global variables or momenta of leptons can be constructed along the same lines with obvious modifications given by the different number of variables and different cross sections.

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# FIGURES

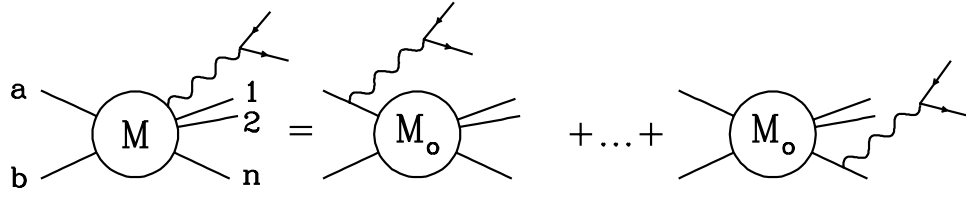


FIG. 1. Matrix element for dilepton production in virtual bremsstrahlung approximation.



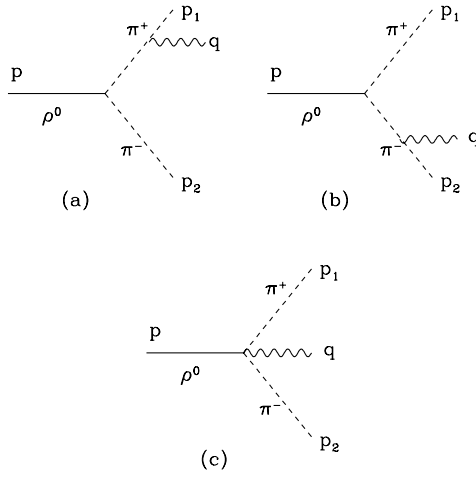


FIG. 2. Feynman diagrams for  $\rho^0 \rightarrow \pi^+ \pi^- \gamma$  decay in the approach of Singer [21].

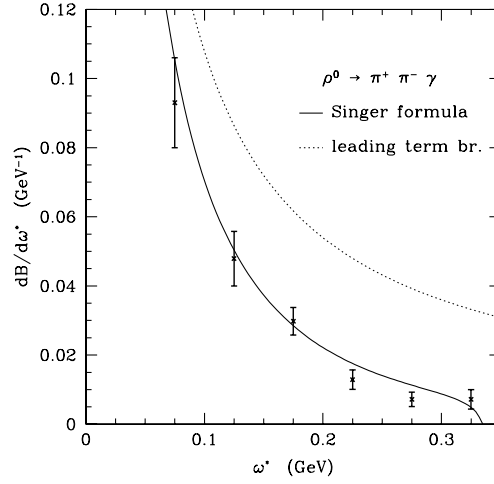


FIG. 3. The differential branching ratio of  $\rho^0 \rightarrow \pi^+ \pi^- \gamma$  as a function of the photon energy in the  $\rho^0$  rest frame. Solid: Singer formula (5.5), dashed: leading term bremsstrahlung formula (6.4). Data [22] were normalized to the integrated branching ratio given in [33].

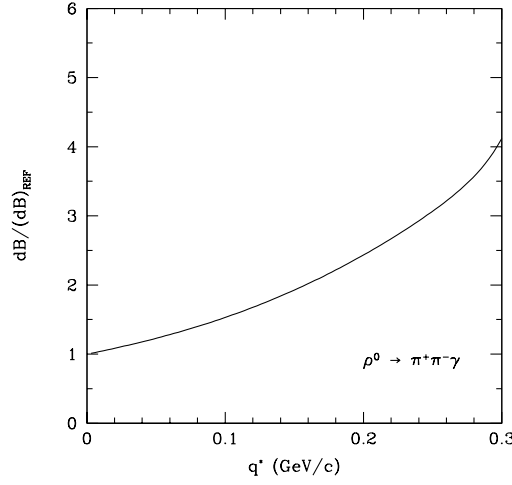


FIG. 4. The ratio of the differential branching ratio for  $\rho^0 \rightarrow \pi^+\pi^-\gamma$  in leading term approximation to that of Singer [21] as a function of the photon momentum in the  $\rho^0$  rest frame.

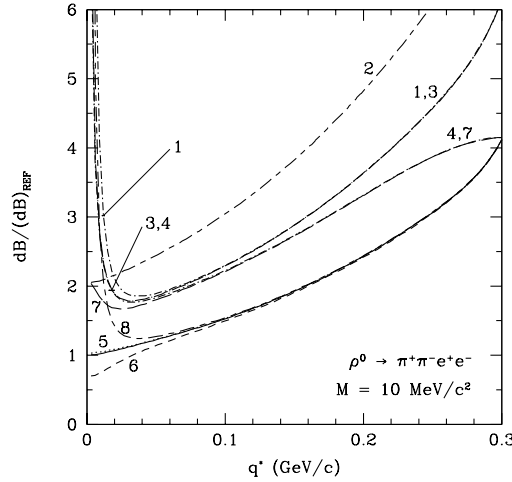


FIG. 5. The ratio of the differential branching ratio for  $\rho^0 \rightarrow \pi^+\pi^-e^+e^-$  calculated from various bremsstrahlung formulas to that of Eq. (5.7) as a function of the dielectron momentum in the  $\rho^0$  rest frame at dielectron mass of  $M = 10 \text{ MeV}/c^2$ . Solid line: formula (6.7); 1: Rückl formula; 2: formula of Goshaw *et al.*; 3: Gale and Kapusta [10] improvement of Rückl formula; 4: Gale and Kapusta [11] improvement of Rückl formula; 5: formula of Balek *et al.*, identical with  $m_T$ -scaling supplemented with the real photon bremsstrahlung formula; 6: formula of Cleymans *et al.*; 7: Haglin *et al.* improvement of Rückl formula; 8: real photon approximation

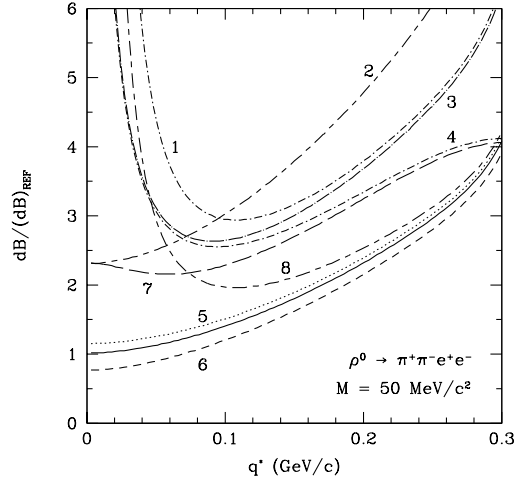


FIG. 6. Same as Fig. 5 but  $M = 50 \text{ MeV}/c^2$ .

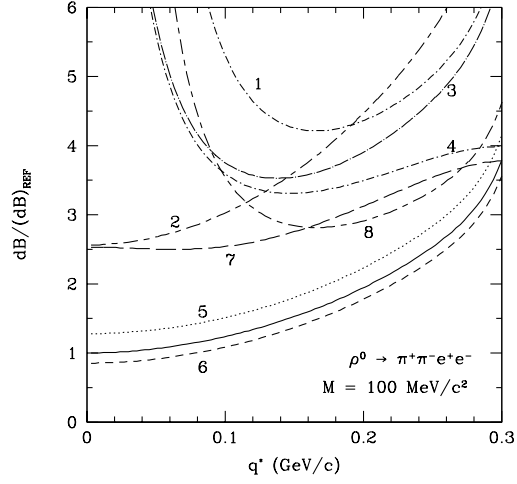


FIG. 7. Same as Fig. 5 but  $M = 100 \text{ MeV}/c^2$ .

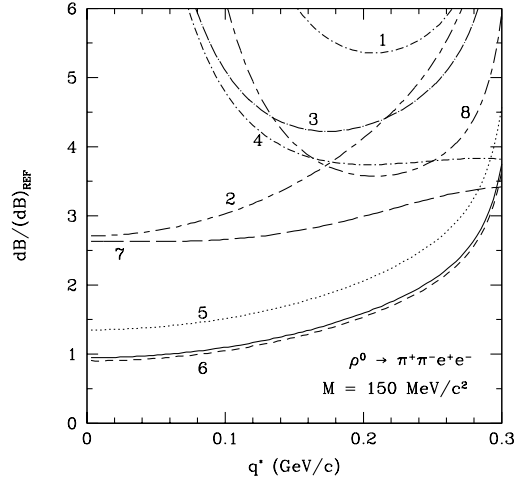


FIG. 8. Same as Fig. 5 but  $M = 150 \text{ MeV}/c^2$ .

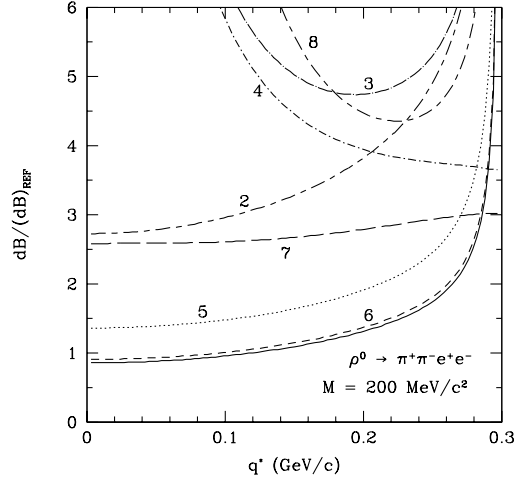


FIG. 9. Same as Fig. 5 but  $M = 200 \text{ MeV}/c^2$ . The Rückl formula curve is off the scale.

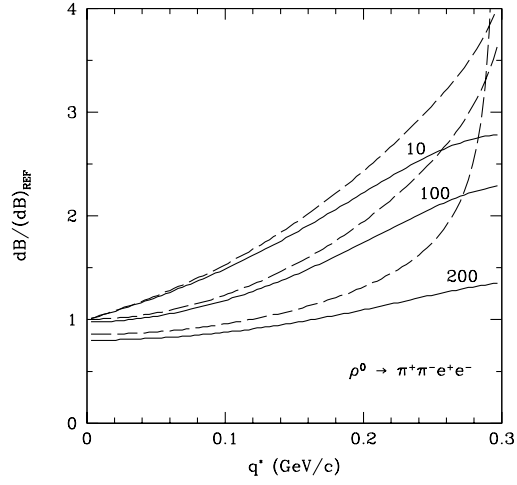


FIG. 10. The role of the phase-space correction factor (4.10) in virtual bremsstrahlung at three different dilepton masses  $M$ : 10 MeV/ $c^2$  (upper), 100 MeV/ $c^2$  (medium), and 200 MeV/ $c^2$  (lower). The leading term bremsstrahlung formula (6.7) with (solid) and without (dashed) the correction factor was divided by (5.10).